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Development of navigation systems in optimal utilization of transportation routes in topology helm graphs using graph labeling

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Abstract: Graph theory plays a vital role in many fields. The graph concept models many relationships and processes in physical, biological, social, transportation, and information systems. One of the essential fields in graph theory is graph labeling. Graph labeling is widely used in various applications such as coding theory, x-ray crystallography, radar, astronomy, circuit design, addressing of communication networks, database management, etc. In this research, we will examine a form of topology of transportation routes based on graph labeling. The topology of the transportation routes that will be studied is a modified of helm graph. This research aims to determine the total vertex irregularity strength of the modified helm graph denoted by H_n for $n \geq 3$. The lower bound is obtained based on the properties of the graph H_n using the existing supporting theorem. The upper bound is obtained by labeling the vertices and edges of the graph H_n in some simple cases. From this labeling, total vertex irregular labeling will be constructed for any n. Based on the results of this research, the total vertex irregularity strength of the graph H_n is $tvs(H_n) = \lceil \frac{3+3n}{4} \rceil$ with $n \geq 3$.

Keywords: *Helm graph, Irregular labeling, Irregular strength, Transportation system.*

1. Introduction

One of the problems encountered in big cities, such as Jakarta, Surabaya, and Makassar, is the increase in the number of vehicles, which needs to be balanced with road infrastructure development. This will lead to the appearance of congestion points on the highway. Various attempts have been made to resolve this problem. One way is to theoretically study transportation networks using graph models, especially labeled graphs.

A labeling of a graph is generally defined as a function of the subsets of elements from G to a set of numbers, typically a set of positive or non-negative integers, which fulfill certain conditions. Several types of graph labeling have been studied, including graceful labeling, magic labeling, anti-magic labeling, and irregular labeling. One good source of information about graph labeling is "A Dynamic Survey of Graph Labeling" [1].

Our research topic is focused on irregular labeling. The concept of irregular labeling on a graph was first introduced by Chartrand, et al. [2]. In Bača, et al. [3] introduced another irregular labeling based on total labeling, namely edge irregular total labeling and vertex total irregular labeling.

Determination of the vertex irregularity strength, the total edge irregularity strength, and the total vertex irregularity strength of a graph can only be done partially for some land transportation network

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models. Several models of transportation networks are still open issues that that need to be resolved entirely.

Our previous research on labeling total irregularities of vertices has been studied for graphs in general. This research produces a lower bound on the total vertex irregularity strength, but the exact value has yet to be obtained. In the same paper, a conjecture is given, which has yet to be resolved [4].

2. Total Vertex Irregular Graph

Historically, Leonhard Euler used graphs to solve the Konigsberg bridge problem in 1736. One of the main points of discussion in graphs is graph labeling. Graph labeling is a function that maps the elements of a graph (vertices or edges) to positive or non-negative integers. If the labeling domain is a set of vertices, it is called point labeling. If the labeling domain is an edge set, then it is called an edge label, and if the labeling domain is a set of vertices and edges, then it is called total labeling [4].

Definition 1. Bac˘a, et al. [3] *The vertex weight v on the total labeling f is the vertex label v added to the sum of all the edges labels incident to v, i.e*

$$wt(v) = f(v) + \sum_{uv\in E} f(uv).$$

Definition 2. Bac˘a, et al. [3]*Let $G(V,E)$ be a simple graph. A total labeling $f: V \cup E \to \{1,2,\dots,k\}$ is a total vertex irregular k-labeling on G if for every two different vertices in V satisfy $wt(x) \neq wt(y)$, where $wt(x) = f(x) + \sum_{xu\in E} f(xu)$.*

Definition 3. Bac˘a, et al. [3] *Let $G(V,E)$ be a simple graph. The total vertex irregularity strength of G, dinoted by $tvs(G)$ is a minimum positive integer number k such that G have a total irregular $k - labelling.$*

Theorem 1. Nurdin [4] *Let $G(V,E)$ be a connected simple graph on n_i vertices of degree i $(i = \delta, \delta + 1, \delta + 2, \dots, \Delta)$, where δ and Δ are minimum and maximum degree on G, respectively. Then*

$$tvs(G) \geq max\left\{\left\lceil\frac{\delta + n_\delta}{\delta + 1}\right\rceil, \left\lceil\frac{\delta + n_\delta + n_{\delta+1}}{\delta + 2}\right\rceil, \dots, \left\lceil\frac{\delta + \sum_{i=\delta}^{\Delta} n_i}{\Delta + 1}\right\rceil\right\}.$$

Several studies have been carried out using total irregular labeling. Aarthi [5], reminded the total vertex irregularity strength in the helm graph Aarthi [5]. Ahmad, et al. [6] determined the total vertex irregularity strength in the composite isomorphic helm graph Ahmad, et al. [6]. Indriati Widodo, et al. [7] determined generalized helm graphs' total vertex irregularity strength [7]. Previously, Nurdin [4] had determined a caterpillar graph's total vertex irregularity strength Nurdin [4]. Likewise, in Hinding, et al. [8] determined the irregularity strength of a diamond graph [8]. In this regard, Siddiqui Nurdin and Baskoro [9] examined the total edge irregularity strength of the disjoint union of the helm graph [9]. However, researchers have yet to determine the total irregularity strength of the modified helm graph vertices. Therefore, this study evaluates the total irregularity strength of the modified helm graph.

3. Results

The graph for which the total vertex irregularity strength will be determined in this paper is the helm g, which is modified so that the vertices are only degrees 3 or 5. Formally, the graph in question is as follows.

Definition 4. *Let H_n be a helm graph. The $\mathcal{H}_n$ is a graph constructed from helm graph H_n by removing the central vertex and adding $2n$ vertices, namely $w_1, v_1, w_2, v_2, \dots, w_n, v_n$ such that*

1. $x_1 w_n$, $x_{i+1} w_i$ *for* $i = 1, 2, \dots, n-1$;
2. $x_n v_1$, $x_{i-1} v_i$ *for* $i = 2, 3, \dots, n$;

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3. $y_iw_i,\ y_iv_i,\ w_iv_i$ *for* $i = 2, 3, \dots, n$

The set of vertices and set of edges of the graph H_n are as follows

$$V(\mathcal{H}_n) = \{x_i, y_i, w_i, v_i \mid i = 1, 2, \dots, n\} \text{ and}$$

$$\begin{aligned} E(\mathcal{H}_n) = \{x_1w_n, x_nv_i\} &\cup \{x_{i+1}w_i \mid i = 1, 2, \dots, n-1\} \\ &\cup \{x_{i-1}v_i \mid i = 2, 3, \dots, n\} \\ &\cup \{x_iy_i, y_iw_i, y_iv_i, w_iv_i \mid i = 1, 2, \dots, n\} \\ &\cup \{x_1x_2, x_2x_3, \dots, x_{n-1}x_n, x_nx_1\}. \end{aligned}$$

In determining the total vertex irregularity strength of the modified helm graph, the lower bound will be analyzed first based on the properties of the graph $\boldsymbol{\mathcal{H}_n}$ and use Theorem 1. Meanwhile, to determine the upper bound, a total vertex irregular labeling will be constructed

Theorem 2. *Let* $\boldsymbol{\mathcal{H}_n}$ *be modified helm graph. For* $\boldsymbol{n \geq 3}$ *and* $\boldsymbol{n}$ *is a positive integer number, then*

$$\boldsymbol{tvs(\mathcal{H}_n) \geq \left\lceil \frac{3+3n}{4} \right\rceil}.$$

Next, to prove that $\boldsymbol{tvs(\mathcal{H}_n) \leq \left\lceil \frac{3+3n}{4} \right\rceil}$, a total vertex irregular labeling is constructed as follows. There are four cases in the construction of a total vertex labeling on $\boldsymbol{\mathcal{H}_n}$.

Case 1. For $n = 4t - 1$, where t is some positive integer. The total vertex labeling f is

$$f(x_1) = 1.$$

$$f(x_2) = s + 1.$$

$$f(x_i) = \begin{cases} \frac{3i+3}{4} + s - i + 1, & i \leq s \\ \frac{3i+3}{4}, & i > s \end{cases},\ i = 3, 7, 11, \dots, n-4.$$

$$f(x_i) = \begin{cases} \frac{3i+4}{4} + s - i + 1, & i \leq s \\ \frac{3i+4}{4}, & i > s \end{cases},\ i = 4, 8, 12, \dots, n-3.$$

$$f(x_i) = \begin{cases} \frac{3i+5}{4} + s - i + 1, & i \leq s \\ \frac{3i+5}{4}, & i > s \end{cases},\ i = 5, 9, 13, \dots, n-2.$$

$$f(x_i) = \begin{cases} \frac{3i+6}{4} + s - i + 1, & i \leq s \\ \frac{3i+6}{4}, & i > s \end{cases},\ i = 6, 10, 14, \dots, n-1.$$

$$f(x_n) = \frac{3n+3}{4}.$$

$$f(y_i) = 1,\ i = 1, 2, \dots, n.$$

$$f(w_1) = 1.$$

$$f(w_2) = \begin{cases} 2, & s = 1 \\ 1, & s = 2, 3, \dots \end{cases}$$

$$f(w_i) = \begin{cases} 1, & i \leq s \\ \frac{3i+3}{4} - s + 1, & i > s \end{cases},\ i = 3, 7, 11, \dots, n-4.$$

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$$f(w_i)=\begin{cases}1, & i\le s\\ \frac{3i+4}{4}-s+1, & i>s\end{cases},\ i=4,8,12,\dots,n-3.$$
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$$f(w_i)=\begin{cases}1, & i\le s\\ \frac{3i+6}{4}-s+1, & i>s\end{cases},\ i=6,10,14,\dots,n-1.$$
$f(w_n)=2s+1.$
$f(v_i)=3s,\ i=1,2,\dots,n.$
$f(x_1x_2)=f(x_2x_3)=f(x_{i-1}x_i)=f(x_ix_{i+1})=f(x_{n-1}x_n)=f(x_nx_1)=3s.$
$f(x_1y_1)=f(x_2y_2)=1.$
$$f(x_iy_i)=\begin{cases}\frac{i+1}{4}, & i=3,7,11,\dots,n-4.\\ \frac{i}{4}, & i=4,8,12,\dots,n-3.\\ \frac{i-1}{4}, & i=5,9,13,\dots,n-2.\\ \frac{i-2}{4}, & i=6,10,14,\dots,n-1.\end{cases}$$
$f(x_ny_n)=\frac{n+1}{4}.$
$f(x_2w_1)=2s.$
$$f(x_3w_2)=\begin{cases}3s-1, & s=1\\ 2s+1, & s=2,3,\dots\end{cases}$$
$$f(x_{i+1}w_i)=\begin{cases}2s+\frac{3i+3}{4}-1, & i\le s\\ 3s-1, & i>s\end{cases},\ i=3,7,11,\dots,n-4.$$
$$f(x_{i+1}w_i)=\begin{cases}2s+\frac{3i+4}{4}-1, & i\le s\\ 3s-1, & i>s\end{cases},\ i=4,8,12,\dots n-3.$$
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$f(x_1w_n)=f(x_nw_{n-1})=\frac{3n+3}{4}-1=3s-1.$
$$f(x_iw_{i-1})=\begin{cases}2s+i-2, & i\le s\\ 3s-1, & i>s\end{cases}$$
$f(x_1v_2)=f(x_iv_{i+1})=f(x_{n-1}v_n)=f(x_nv_1)=3s.$
$f(y_iw_i)=1,\ i=1,2,\dots,n.$
$f(y_1v_1)=1.$
$f(y_2v_2)=2.$

$$f(y_i v_i) = \begin{cases} \frac{3i+3}{4}, & i = 3, 7, 11, \dots, n-4, \\ \frac{3i+4}{4}, & i = 4, 8, 12, \dots, n-3, \\ \frac{3i+5}{4}, & i = 5, 9, 13, \dots, n-2, \\ \frac{3i+6}{4}, & i = 6, 10, 14, \dots, n-1, \end{cases}$$

$$f(y_n v_n) = \frac{3n+3}{4}.$$

$$f(w_1 v_1) = f(w_2 v_2) = 2s + 1.$$

$$f(w_i v_i) = \begin{cases} \frac{i+1}{4} + 2s, & i = 3, 7, 11, \dots, n-4, \\ \frac{i}{4} + 2s, & i = 4, 8, 12, \dots, n-3, \\ \frac{i-1}{4} + 2s, & i = 5, 9, 13, \dots, n-2, \\ \frac{i-2}{4} + 2s, & i = 6, 10, 14, \dots, n-1, \end{cases}$$

$$f(w_n v_n) = \frac{n+1}{4} + 2s.$$

Based on this function, it can be shown that all vertices weights are different and the maximum value of the function is $\left\lceil \frac{\mathbf{3+3}\boldsymbol{n}}{\mathbf{4}} \right\rceil$. This shows that $f: V \cup E \to \left\{1, 2, 3, \cdots, \left\lceil \frac{\mathbf{3+3}\boldsymbol{n}}{\mathbf{4}} \right\rceil \right\}$ is a function a total vertex irregular $k - labelling$ where $k = \left\lceil \frac{\mathbf{3+3}\boldsymbol{n}}{\mathbf{4}} \right\rceil$. Thus it is obtained that $\boldsymbol{tvs}(\boldsymbol{\mathcal{H}_n}) \leq \left\lceil \frac{\mathbf{3+3}\boldsymbol{n}}{\mathbf{4}} \right\rceil$ for $n = 4t - 1$, where t is some positive integer.

Case 2. For $n = 4t$, where t is some positive integer. The total vertex labeling g is

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$$g(x_n) = \frac{3n+4}{4}.$$

$$g(y_i) = 1, \ i = 1, 2, \dots, n.$$

$$g(w_1) = 1.$$

$$g(w_2) = \begin{cases} 2, & s = 1 \\ 1, & s = 2, 3, \dots \end{cases}$$

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$$g(w_i)=\begin{cases}1, & i\le s\\ \frac{3i+3}{4}-s+1, & i>s\end{cases},\ i=3,7,11,\dots,n-1.$$
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$$g(w_n)=2s+2.$$
$$g(v_i)=3s+1,\ i=1,2,\dots,n.$$
$$g(x_1x_2)=g(x_{n-1}x_n)=3s.$$
$$g(x_2x_3)=g(x_nx_1)=3s+1.$$
$$g(x_ix_{i+1})=\begin{cases}3s, & \begin{cases}i=3,7,11,\dots,n-1.\\ i=5,9,13,\dots n-3.\end{cases}\\ 3s+1, & \begin{cases}i=4,8,12,\dots,n-4.\\ i=6,10,14,\dots,n-2.\end{cases}\end{cases}$$
$$g(x_{i-1}x_i)=\begin{cases}3s+1, & \begin{cases}i=3,7,11,\dots,n-1.\\ i=5,9,13,\dots n-3.\end{cases}\\ 3s, & \begin{cases}i=4,8,12,\dots,n-4.\\ i=6,10,14,\dots,n-2.\end{cases}\end{cases}$$
$$g(x_1y_1)=f(x_2y_2)=1.$$
$$g(x_iy_i)=\begin{cases}\frac{i+1}{4}, & i=3,7,11,\dots,n-1.\\ \frac{i}{4}, & i=4,8,12,\dots,n-4.\\ \frac{i-1}{4}, & i=5,9,13,\dots,n-3.\\ \frac{i-2}{4}, & i=6,10,14,\dots,n-2.\end{cases}$$
$$g(x_ny_n)=\frac{n}{4}.$$
$$g(x_2w_1)=2s+1.$$
$$g(x_3w_2)=\begin{cases}3s, & s=1\\ 2s+2, & s=2,3,\dots\end{cases}$$
$$g(x_{i+1}w_i)=\begin{cases}2s+\frac{3i+3}{4}, & i\le s\\ 3s, & i>s\end{cases},\ i=3,7,11,\dots,n-1.$$
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$$g(x_1w_n)=g(x_nw_{n-1})=\frac{3n+4}{4}-1=3s.$$

$$g(x_i w_{i-1}) = \begin{cases} 2s + i - 1, & i \le s \\ 3s, & i > s \end{cases}$$
$$g(x_1 v_2) = g(x_i v_{i+1}) = g(x_{n-1} v_n) = g(x_n v_1) = 3s + 1$$
$$g(y_i w_i) = 1, \ i = 1, 2, \dots, n.$$
$$g(y_1 v_1) = 1.$$
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$$g(y_n v_n) = \frac{3n+4}{4}.$$
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$$g(w_i v_i) = \begin{cases} \frac{i+1}{4} + 2s, & i = 3, 7, 11, \dots, n-1, \\ \frac{i}{4} + 2s, & i = 4, 8, 12, \dots, n-4, \\ \frac{i-1}{4} + 2s, & i = 5, 9, 13, \dots, n-3, \\ \frac{i-2}{4} + 2s, & i = 6, 10, 14, \dots, n-2, \end{cases}$$
$$g(w_n v_n) = \frac{n}{4} + 2s.$$

Based on this function, it can be shown that all vertices weights are different and the maximum value of the function is $\left\lceil \frac{\mathbf{3+3n}}{\mathbf{4}} \right\rceil$. This shows that $g: V \cup E \to \left\{1, 2, 3, \cdots, \left\lceil \frac{\mathbf{3+3n}}{\mathbf{4}} \right\rceil \right\}$ is a function a total vertex irregular $k - labelling$ where $k = \left\lceil \frac{\mathbf{3+3n}}{\mathbf{4}} \right\rceil$. Thus it is obtained that $\boldsymbol{tvs(\mathcal{H}_n)} \le \left\lceil \frac{\mathbf{3+3n}}{\mathbf{4}} \right\rceil$ for $\boldsymbol{n = 4t}$, where $\boldsymbol{t}$ is some positive integer.

Case 3. For $n = 4t + 1$, where t is some positive integer. The total vertex labeling h is

$$h(x_1) = 1.$$
$$h(x_2) = s + 1.$$
$$h(x_i) = \begin{cases} \frac{3i+3}{4} + s - i + 1, & i \le s \\ \frac{3i+3}{4}, & i > s \end{cases}, \ i = 3, 7, 11, \dots, n-2.$$
$$h(x_i) = \begin{cases} \frac{3i+4}{4} + s - i + 1, & i \le s \\ \frac{3i+4}{4}, & i > s \end{cases}, \ i = 4, 8, 12, \dots, n-1.$$
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$$h(x_i) = \begin{cases} \frac{3i+6}{4} + s - i + 1, & i \le s \\ \frac{3i+6}{4}, & i > s \end{cases}, \ i = 6, 10, 14, \dots, n-3.$$
$$h(x_n) = \frac{3n+5}{4}.$$

$h(y_i) = 1, \; i = 1, 2, \dots, n.$
$h(w_1) = 1.$
$h(w_2) = \begin{cases} 2, & s = 1 \\ 1, & s = 2, 3, \dots \end{cases}$
$h(w_i) = \begin{cases} 1, & i \le s \\ \frac{3i+3}{4} - s + 1, & i > s \end{cases}, \; i = 3, 7, 11, \dots, n-2.$
$h(w_i) = \begin{cases} 1, & i \le s \\ \frac{3i+4}{4} - s + 1, & i > s \end{cases}, \; i = 4, 8, 12, \dots, n-1.$
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$h(w_n) = 2s + 3.$
$h(v_i) = 3s + 2, \; i = 1, 2, \dots, n.$
$h(x_1x_2) = h(x_2x_3) = h(x_{i-1}x_i) = h(x_ix_{i+1}) = h(x_{n-1}x_n) = h(x_nx_1) =$
$3s + 1.$
$h(x_1y_1) = h(x_2y_2) = 1.$
$h(x_iy_i) = \begin{cases} \frac{i+1}{4}, & i = 3, 7, 11, \dots, n-2. \\ \frac{i}{4}, & i = 4, 8, 12, \dots, n-1. \\ \frac{i-1}{4}, & i = 5, 9, 13, \dots, n-4. \\ \frac{i-2}{4}, & i = 6, 10, 14, \dots, n-3. \end{cases}$
$h(x_ny_n) = \frac{n-1}{4}.$
$h(x_2w_1) = 2s + 2.$
$h(x_3w_2) = \begin{cases} 3s + 1, & s = 1 \\ 2s + 3, & s = 2, 3, \dots \end{cases}$
$h(x_{i+1}w_i) = \begin{cases} 2s + \frac{3i+3}{4} + 1, & i \le s \\ 3s + 1, & i > s \end{cases}, \; i = 3, 7, 11, \dots, n-2.$
$h(x_{i+1}w_i) = \begin{cases} 2s + \frac{3i+4}{4} + 1, & i \le s \\ 3s + 1, & i > s \end{cases}, \; i = 4, 8, 12, \dots, n-1.$
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$h(x_1w_n) = h(x_nw_{n-1}) = \frac{3n+5}{4} - 1 = 3s + 1.$
$h(x_iw_{i-1}) = \begin{cases} 2s + i, & i \le s \\ 3s + 1, & i > s \end{cases}$
$h(x_1v_2) = h(x_iv_{i+1}) = h(x_{n-1}v_n) = h(x_nv_1) = 3s + 2.$
$h(y_iw_i) = 1, \; i = 1, 2, \dots, n.$
$h(y_1v_1) = 1.$

$h(y_2v_2) = 2.$

$$h(y_iv_i) = \begin{cases} \frac{3i+3}{4}, & i = 3,7,11,\dots,n-2, \\ \frac{3i+4}{4}, & i = 4,8,12,\dots,n-1, \\ \frac{3i+5}{4}, & i = 5,9,13,\dots,n-4, \\ \frac{3i+6}{4}, & i = 6,10,14,\dots,n-3, \end{cases}$$

$h(y_nv_n) = \frac{3n+5}{4}.$

$h(w_1v_1) = h(w_2v_2) = 2s + 1.$

$$h(w_iv_i) = \begin{cases} \frac{i+1}{4} + 2s, & i = 3,7,11,\dots,n-2, \\ \frac{i}{4} + 2s, & i = 4,8,12,\dots,n-1, \\ \frac{i-1}{4} + 2s, & i = 5,9,13,\dots,n-4, \\ \frac{i-2}{4} + 2s, & i = 6,10,14,\dots,n-3, \end{cases}$$

$$h(w_nv_n) = \frac{n-1}{4} + 2s$$

Based on this function, it can be shown that all vertices weights are different and the maximum value of the function is $\left\lceil \frac{\mathbf{3+3n}}{\mathbf{4}} \right\rceil$. This shows that $h: V \cup E \to \left\{1,2,3,\cdots,\left\lceil \frac{\mathbf{3+3n}}{\mathbf{4}} \right\rceil\right\}$ is a function a total vertex irregular $k-labelling$ where $k = \left\lceil \frac{\mathbf{3+3n}}{\mathbf{4}} \right\rceil$. Thus it is obtained that $\boldsymbol{tvs(\mathcal{H}_n)} \le \left\lceil \frac{\mathbf{3+3n}}{\mathbf{4}} \right\rceil$ for $n = 4t + 1,$ where t is some positive integer.

Case 4. For $n = 4t + 2,$ where t is some positive integer. The total vertex labeling q is

$q(x_1) = 1.$

$q(x_2) = s + 1.$

$$q(x_i) = \begin{cases} \frac{3i+3}{4} + s - i + 1, & i \le s \\ \frac{3i+3}{4}, & i > s \end{cases}, \ i = 3,7,11,\dots,n-3.$$

$$q(x_i) = \begin{cases} \frac{3i+4}{4} + s - i + 1, & i \le s \\ \frac{3i+4}{4}, & i > s \end{cases}, \ i = 4,8,12,\dots,n-2.$$

$$q(x_i) = \begin{cases} \frac{3i+5}{4} + s - i + 1, & i \le s \\ \frac{3i+5}{4}, & i > s \end{cases}, \ i = 5,9,13,\dots,n-1.$$

$$q(x_i) = \begin{cases} \frac{3i+6}{4} + s - i + 1, & i \le s \\ \frac{3i+6}{4}, & i > s \end{cases}, \ i = 6,10,14,\dots,n-4.$$

$q(x_n) = \frac{3n+6}{4}.$

$q(y_i) = 1, \ i = 1,2,\dots,n.$

$q(w_1) = 1.$

$$q(w_2) = \begin{cases} 2, & s = 1 \\ 1, & s = 2,3,\dots \end{cases}$$

$$q(w_i)=\begin{cases}1, & i\le s\\ \frac{3i+3}{4}-s+1, & i>s\end{cases},\ i=3,7,11,\dots,n-3.$$
$$q(w_i)=\begin{cases}1, & i\le s\\ \frac{3i+4}{4}-s+1, & i>s\end{cases},\ i=4,8,12,\dots,n-2.$$
$$q(w_i)=\begin{cases}1, & i\le s\\ \frac{3i+5}{4}-s+1, & i>s\end{cases},\ i=5,9,13,\dots,n-1.$$
$$q(w_i)=\begin{cases}1, & i\le s\\ \frac{3i+6}{4}-s+1, & i>s\end{cases},\ i=6,10,14,\dots,n-4.$$
$$q(w_n)=2s+4.$$
$$q(v_i)=3s+3,\ i=1,2,\dots,n.$$
$$q(x_1x_2)=q(x_{n-1}x_n)=3s+1.$$
$$q(x_2x_3)=q(x_nx_1)=3s+2.$$
$$q(x_ix_{i+1})=\begin{cases}3s+1, & \begin{cases}i=3,7,11,\dots,n-3.\\ i=5,9,13,\dots n-1.\end{cases}\\ 3s+2, & \begin{cases}i=4,8,12,\dots,n-2.\\ i=6,10,14,\dots,n-4.\end{cases}\end{cases}$$
$$q(x_{i-1}x_i)=\begin{cases}3s+2, & \begin{cases}i=3,7,11,\dots,n-3.\\ i=5,9,13,\dots n-1.\end{cases}\\ 3s+1, & \begin{cases}i=4,8,12,\dots,n-2.\\ i=6,10,14,\dots,n-4.\end{cases}\end{cases}$$
$$q(x_1y_1)=q(x_2y_2)=1.$$
$$q(x_iy_i)=\begin{cases}\frac{i+1}{4}, & i=3,7,11,\dots,n-3.\\ \frac{i}{4}, & i=4,8,12,\dots,n-2.\\ \frac{i-1}{4}, & i=5,9,13,\dots,n-1.\\ \frac{i-2}{4}, & i=6,10,14,\dots,n-4.\end{cases}$$
$$q(x_ny_n)=\frac{n-2}{4}.$$
$$q(x_2w_1)=2s+3.$$
$$q(x_3w_2)=\begin{cases}3s+2, & s=1\\ 2s+4, & s=2,3,\dots\end{cases}$$
$$q(x_{i+1}w_i)=\begin{cases}2s+\frac{3i+3}{4}+2, & i\le s\\ 3s+2, & i>s\end{cases},\ i=3,7,11,\dots,n-3.$$
$$q(x_{i+1}w_i)=\begin{cases}2s+\frac{3i+4}{4}+2, & i\le s\\ 3s+2, & i>s\end{cases},\ i=4,8,12,\dots,n-2.$$
$$q(x_{i+1}w_i)=\begin{cases}2s+\frac{3i+5}{4}+2, & i\le s\\ 3s+2, & i>s\end{cases},\ i=5,9,13,\dots,n-1.$$
$$q(x_{i+1}w_i)=\begin{cases}2s+\frac{3i+6}{4}+2, & i\le s\\ 3s+2, & i>s\end{cases},\ i=6,10,14,\dots,n-4.$$
$$q(x_1w_n)=q(x_nw_{n-1})=\frac{3n+6}{4}-1=3s+2.$$

$$q(x_i w_{i-1}) = \begin{cases} 2s + i + 1, & i \le s \\ 3s + 2, & i > s \end{cases}$$
$$q(x_1 v_2) = q(x_i v_{i+1}) = q(x_{n-1} v_n) = q(x_n v_1) = 3s + 3.$$
$$q(y_i w_i) = 1, \ \ i = 1, 2, \dots, n.$$
$$q(y_1 v_1) = 1.$$
$$q(y_2 v_2) = 2.$$
$$q(y_i v_i) = \begin{cases} \frac{3i+3}{4}, & i = 3, 7, 11, \dots, n-3, \\ \frac{3i+4}{4}, & i = 4, 8, 12, \dots, n-2, \\ \frac{3i+5}{4}, & i = 5, 9, 13, \dots, n-1, \\ \frac{3i+6}{4}, & i = 6, 10, 14, \dots, n-4, \end{cases}$$
$$q(y_n v_n) = \frac{3n+6}{4}.$$
$$q(w_1 v_1) = q(w_2 v_2) = 2s + 1.$$
$$q(w_i v_i) = \begin{cases} \frac{i+1}{4} + 2s, & i = 3, 7, 11, \dots, n-3, \\ \frac{i}{4} + 2s, & i = 4, 8, 12, \dots, n-2, \\ \frac{i-1}{4} + 2s, & i = 5, 9, 13, \dots, n-1, \\ \frac{i-2}{4} + 2s, & i = 6, 10, 14, \dots, n-4, \end{cases}$$
$$q(w_n v_n) = \frac{n-2}{4} + 2s.$$

Based on this function, it can be shown that all vertices weights are different and the maximum value of the function is $\left\lceil \frac{\mathbf{3+3n}}{\mathbf{4}} \right\rceil$. This shows that $q: V \cup E \to \left\{1, 2, 3, \cdots, \left\lceil \frac{\mathbf{3+3n}}{\mathbf{4}} \right\rceil \right\}$ is a function a total vertex irregular $k - labelling$ where $k = \left\lceil \frac{\mathbf{3+3n}}{\mathbf{4}} \right\rceil$. Thus it is obtained that $\boldsymbol{tvs(\mathcal{H}_n)} \le \left\lceil \frac{\mathbf{3+3n}}{\mathbf{4}} \right\rceil$ for $n = 4t + 2$, where t is some positive integer. Based on these four cases, it is concluded that $tvs(\mathcal{H}_n) = \left\lceil \frac{3+3n}{4} \right\rceil$ for $n \ge 3$ and n is a positive integer number.

4. Conclusion

This research contributes significantly to understanding transportation route optimization through the lens of graph theory, specifically focusing on graph labeling within modified helm graph topologies. By applying irregular labeling, this study explores the vertex irregularity strength required for efficient transportation network configurations, which is critical in addressing the increasing congestion in urban transportation systems. The findings establish the lower and upper bounds of vertex irregularity strength within these graph structures, providing foundational parameters for further modeling and practical application in real-world transportation networks. These insights enhance theoretical perspectives in graph theory and offer a pathway for developing algorithms that can be used to optimize route navigation in complex transportation systems.

Transparency:

The authors confirm that the manuscript is an honest, accurate, and transparent account of the study; that no vital features of the study have been omitted; and that any discrepancies from the study as planned have been explained. This study followed all ethical practices during writing.

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References

[1] J. A. Gallian, "A dynamic survey of graph labelling," *The Electronic Journal of Combinatorics*, vol. 6, no. 25, pp. 4 – 623, 2024.

[2] G. Chartrand, M. S. Jacobson, J. Lehel, O. R. Oellermann, S. Ruiz, and F. Saba, "Irregular network," *Congruent Numbers*, vol. 64, pp. 187-192, 1988.

[3] M. Bac ̌a, S. Jendrol, M. Miller, and J. Ryan, "On irregular total labeling," *Discrete Mathematics*, vol. 307, pp. 1378–1388, 2007. https://doi.org/10.1016/j.disc.2005.11.075

[4] M. Z. Nurdin, "Firman,"Vertex-irregular labeling and vertex-irregular total labeling on caterpillar graph," *International Journal of Applied Mathematics and Statistics*, vol. 40, no. 10, pp. 99-105, 2013.

[5] S. Aarthi, "Stability of total irregularity strength of helm graphs," *IJESC*, vol. 7, no. 8, 2017.

[6] A. Ahmad, E. Baskoro, and M. Imran, "Total vertex irregularity strength of disjoint union of Helm graphs," *Discussiones Mathematicae Graph Theory*, vol. 32, no. 3, pp. 427-434, 2012. https://doi.org/10.7151/dmgt.1619

[7] D. Indriati Widodo, I. E. Wijayanti, K. A. Sugeng, M. Bača, and A. S. Fenovcikova, "The total vertex irregularity strength of generalized helm graphs and prisms with outer pendant edges," *Australian Hair Comb*, vol. 65, no. 1, pp. 14–26, 2016.

[8] N. Hinding, D. Firmayasari, H. Basir, M. Bača, and A. Semaničová-Feňovčíková, "On irregularity strength of diamond network," *AKCE International Journal of Graphs and Combinatorics*, vol. 15, no. 3, pp. 291-297, 2018. https://doi.org/10.1016/j.akcej.2017.10.003

[9] M. K. Siddiqui Nurdin and E. T. Baskoro, "Total edge irregularity strength of the disjoint union of helm graph," *Journal of Mathematical and Fundamental Sciences*, vol. 42, no. 2, pp. 163–171, 2013.